ON SOME PROPERTIES OF SQUEEZING FUNCTIONS OF BOUNDED DOMAINS

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ABSTRACT. The main purpose of the present paper is to introduce the notion of squeezing functions of bounded domains and study some properties of them. The relation to geometric and analytic structures of bounded domains will be investigated. Existence of related extremal maps and continuity of squeezing functions are proved. Holomorphic homogeneous regular domains are exactly domains whose squeezing functions have positive lower bounds. Completeness of certain intrinsic metrics and pseudoconvexity of holomorphic homogeneous regular domains are proved by alternative method. In dimension one case, we get a neat description of boundary behavior of squeezing functions of finitely connected planar domains. This leads to a necessary and sufficient conditions for a finitely connected planar domain to be a holomorphic homogeneous regular domain. Consequently, we can recover some important results in complex analysis. For annuli, we obtain some interesting properties of their squeezing functions. We finally exhibit some examples of bounded domains whose squeezing functions can be given explicitly.

1. INTRODUCTION

Bounded domains are elementary objects of study in complex analysis. To study complex and geometric structures of bounded domains, one may consider holomorphic maps form bounded domains to some standard domains such as balls and vice visa. The basic idea goes back to Carathéodory, and a typical example is the definition of Carathéodory metric and Kobayashi metric. Holomorphic maps from bounded domains to the unit ball with certain extremal properties are called Carathéodory maps, which can be explicitly given for some special domains such as bounded symmetric domains [11][13] and ellipsoids [17]. Recently, by considering embeddings of general bounded domains into the unit ball, a new concept of holomorphic homogeneous regular domains was introduced in [14]. Holomorphic homogeneous regular domains are generalizations of Teichmüller spaces, and they admit some nice geometric and analytic properties (see [14][21]).

Motivated by the works mentioned above, especially that in [14], we introduce the notion of squeezing functions defined on general bounded domains as follows:

Definition 1.1. Let D be a bounded domain in $\mathbb{C}^n$. For $p \in D$ and an (open) holomorphic embedding $f : D \to B^n$ with $f(p) = 0$, we define

$$s_D(p, f) = \sup\{r | B^n(0, r) \subset f(D)\},$$

and the squeezing number $s_D(p)$ of D at p is defined as

$$s_D(p) = \sup_f \{s_D(p, f)\},$$

where the supremum is taken over all holomorphic embeddings $f : D \to B^n$ with $f(p) = 0$, B^n is the unit ball in $\mathbb{C}^n$ and $B^n(0, r)$ is the ball in $\mathbb{C}^n$ with center 0 and radius r. We call s_D the *squeezing function* on D.

By definition, it is clear that squeezing functions are invariant under biholomorphic transformations, so they can be viewed as a kind of holomorphic invariants of bounded domains. The main purpose of the present paper is to investigate some properties of squeezing functions and their relations with geometric and analytic structures of bounded domains.

Squeezing functions are always positive and bounded above by 1. It is interesting to estimate their lower and upper bounds, which are numerical holomorphic invariants of bounded domains, by the holomorphic invariance of squeezing functions. Holomorphic homogeneous regular domains defined in [14] are exactly bounded domains whose squeezing functions admit positive lower bounds. They contain some interesting objects such as bounded homogeneous domains, Teichmüller spaces, bounded domains covering compact Kähler manifolds and strictly convex domains with C^2-boundary [21].

It is easy to see that the squeezing function of the unit ball B^n is constant with value 1. A natural question is whether the squeezing function of a bounded domain D can attain the value 1 at some point x in D if D is not holomorphic equivalent to B^n? This question is related to the existence of an extremal map which realizes the supremum $s_D(x)$, i.e., the existence of a holomorphic embedding $f : D \to B^n$ such that $f(x) = 0$ and $B^n(0, s_D(x)) \subset f(D)$. We will prove the existence of extremal maps, by using a higher dimensional generalization of Hurwitz theorem that will be proved as well in the present article. As a consequence, S_D attains the value 1 in D if and only if D is holomorphic equivalent to the unit ball. On the other hand, as we will see, there exist domains whose squeezing functions have supremum 1, but they are not holomorphic equivalent to the unit ball.

An elementary property of regularity of squeezing functions is their continuity. It can be proved by using the decreasing property of Kobayashi metrics. From the continuity property, one can see that a bounded domain is a holomorphic homogeneous regular domain if it covers a compact complex manifold.

Properties of squeezing functions can reflect some geometric and analytic properties of bounded domains. Boundary behavior of squeezing functions implies certain boundary estimate of Carathéodory metric, which implies completeness of the metric in some special cases. For a bounded domain whose squeezing function admits a positive lower bound, i.e. a holomorphic homogeneous regular domain, it is known that the intrinsic metrics – the Carathéodory metric, the Kobayashi metric and the Bergman metric – on it are equivalent [14], and they are all complete [21]. We will prove the completeness of these metrics by alternative method based on Lu's result on comparsion of Bergman metric and Carathéodory metric [16]. A result in several complex variables states that completeness of the Carathéodory metric of a domain implies its pseudoconvexity (see e.g. [8]), hence a holomorphic homogeneous regular domain must be a pseudoconvex domain.

Squeezing functions of planar domains have nice properties. For finitely connected planar domains, we get a neat description of the boundary behavior of their squeezing functions. As a result, we get the necessary and sufficient condition for such a domain to be a holomorphic homogeneous regular domain. Surprisingly,

the squeezing function s_D of a bounded planar domain D with smooth boundary admits the boundary behavior

$$\lim_{z \to \partial D} s_D(z) = 1. \tag{1}$$

By the continuity of squeezing functions, equality (1) implies that all smoothly bounded planar domains are holomorphic homogeneous regular domains. As a result, we can recover some important results about planar domains, e.g., the three intrinsic metrics mentioned above on a bounded planar domain with smooth boundary are all complete, and they are equivalent, and a smoothly bounded planar domain must be hyperconvex, i.e., it admits a bounded exhaustive subharmonic function. In particular, equality (1) also implies that s_D can be extended continuously to $\overline{D}$ for a planar domain D with smooth boundary. We don't know whether this is true in general case, i.e., whether s_D can be extended continuously to $\overline{D}$ for all bounded domains $D \subset \mathbb{C}^n$ with smooth boundary.

It is clear that the product of two holomorphic homogeneous regular domains is still a holomorphic homogeneous regular domain, so smoothly bounded planar domains and their products provide a class of holomorphic homogeneous regular domains which are generally not contained in the list of holomorphic homogeneous regular domains mentioned above. As remarked in [14], it may be interesting to investigate whether the Kobayashi metric and the Carathéodory metric on a Teichmüller space coincide or not; on the other hand, the Kobayashi metric and the Carathéodory metric on general holomorphic homogeneous regular domains constructed here don't coincide.

The simplest nontrivial smoothly bounded planar domains are annuli. However, even in this special case, squeezing functions admit nontrivial properties. With certain investigation, we conjecture that the conformal structure of an annulus is characterized by the exact lower bound of its squeezing function.

The squeezing functions can be given explicitly for classical bounded symmetric domains. In this special case, we see that the extremal maps for squeezing functions defined as above can be given by the Carathéodory maps (see §7 for exact definition). However, this does not hold for general domains such as annuli. In fact, the Carathéodory maps (often called Ahlfors maps for planar domains) of a bounded planar domain can not even be injective if the domain is not simply connected (see e.g. [5]). It seems that the obstruction for the coincidence of the two types of extremal maps comes from topology. Therefore, we conjecture that the extremal maps of a contractible domain are given by Carathéodory maps.

The rest of this article is organized as follows. In §2, we generalize the Hurwitz theorem from one complex variable analysis to several complex variables, and use this generalization to establish the existence of extremal maps that are defined as above; In §3, we prove the continuity of squeezing functions of general bounded domains; In §4, we give a boundary estimate of Carathéodory metrics in term of boundary behavior of squeezing functions, and prove the completeness of certain intrinsic metrics on holomorphic homogeneous regular domains , as a corollary, we get the pseudoconvexity of these domains; In §5, we study squeezing functions on planar domains, and prove equality (1) of smoothly bounded planar domains, we also construct a class of planar holomorphic homogeneous regular domains which are infinitely connected; In §6, we focus on squeezing functions on annuli, and in

the final §7, we give some examples of bounded domains whose squeezing functions can be given explicitly.

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2. Generalized Hurwitz Theorem and the existence of extremal functions

The main aim of this section is to establish the existence of extremal maps related to squeezing functions, i.e., the following

Theorem 2.1. *Let D be a bounded domain in $\mathbb{C}^n$, then for any $x \in D$, there exists a holomorphic embedding $f : D \to B^n$ such that $f(x) = 0$ and $B^n(0, s_D(x)) \subset f(D)$.*

By definition we have $s_D(z) \leq 1$ for all $z \in D$. By Theorem 2.1, we see that $s_D(z) = 1$ for some $z \in D$ if and only if D is holomorphically equivalent to B^n.

To prove Theorem 2.1, we need to generalize Hurwitz's theorem in classical complex analysis to several complex variables. Hurwitz's theorem in one complex variable says that the limit of a sequence of univalent functions on a planar domain is univalent unless it is constant (see e.g. [18]). Of course there is no direct generalization of this result in higher dimensions, however, a modified version described in the following theorem still holds:

Theorem 2.2. *Let D be a bounded domain in $\mathbb{C}^n$ and $x \in D$, let f_i be a sequence of injective holomorphic maps from D to $\mathbb{C}^n$ such that $f_i(x) = 0 \in \mathbb{C}^n$ for all i. Suppose f_i converges to a map $f : D \to \mathbb{C}^n$ uniformly on compact subsets of D. If there exists a neighborhood U of 0 in $\mathbb{C}^n$ such that $U \subset f_i(D)$ for all i, then f is injective.*

To prove Theorem 2.2, we need two lemmas. The first is

Lemma 2.3. *Let D be a domain in $\mathbb{C}^n$ and φ_i a sequence of holomorphic maps from D to $\mathbb{C}^n$ which is convergent to a map $\varphi : D \to \mathbb{C}^n$ uniformly on compact subsets of D. If all φ_i have no zero in D, then φ has no zero in D unless it is identically zero.*

Proof. By the identity theorem of holomorphic functions, we may assume D is a ball. Assume there exists $x \in D$ such that $\varphi(x) = 0$. For any point $z \in D$, consider the intersection of D and the complex line containing x and z, then a version of the classical Hurwitz theorem (see Corollary in P. 162 in [18]) implies that $\varphi(z) = 0$. So φ is identically zero on D. □

The second lemma we need in the proof of Theorem 2.2 is the generalized Rouché's theorem in higher dimensions, whose proof relies on the mapping degree theory in differential topology.

Lemma 2.4. *(Theorem 3 in [15]) Let D be a bounded domain in $\mathbb{C}^n$, suppose f and g are two holomorphic maps from D to $\mathbb{C}^n$ such that*

$$\| g(z) \| < \| f(z) \|, \quad z \in \partial D$$

Then f and $f + g$ have the same number of zeros in D, counting multiplicities, where $\| \cdot \|$ is the standard norm in $\mathbb{C}^n$.

With the above two lemmas, we can give the proof of Theorem 2.2 as follows:

Proof. (Proof of Theorem 2.2)Let $g_i = f_i^{-1}|_U$. By Montel theorem, the sequence $\{g_i\}$ is convergent to a holomorphic map $g : U \to \mathbb{C}^n$ uniformly on compact subsets of U. Note that $g(0) = x$ is a interior point of D, we can assume $g(U) \subset D$ by taking U smaller enough. It is clear that $f_i g_i = Id_U$ for all i. Let i tends to ∞, we get $fg = Id_U$. This implies that the Jacobian determinant $det J_f(x)$ of f at x is not zero. Since f_i are injective, $det J_{f_i}(z) \neq 0$ for all i and all $z \in D$ (see e.g. Theorem 8.5 in [6]). Note that $det J_{f_i}$ converges to $det J_f$ uniformly on compact subsets of D, by Lemma 2.3, we see that $det J_f(z) \neq 0$ for all $z \in D$. We prove f is injective. If it is not the case, there exist $z_1, z_2 \in D$, $z_1 \neq z_2$, such that $f(z_1) = f(z_2)$. Since $det J_f \neq 0$, we can choose a neighborhood $\Omega \subset\subset D$ of z_1 such that $f|_{\overline{\Omega}}$ is injective and $z_2 \notin \overline{\Omega}$. Set $\tilde{f}_i = f_i - f_i(z_1)$ and $\tilde{f} = f - f(z_1)$, then $\tilde{f}_i$ converges to $\tilde{f}$ uniformly on $\overline{\Omega}$. Note that $\tilde{f}$ has a zero in Ω and $\tilde{f}^{-1}(0) \cap \partial\Omega = \emptyset$, by Lemma 2.4, $\tilde{f}_i$ has a zero in Ω for i large enough, which contradicts to the fact that f_i are all injective on D. $\qquad\square$

Remark 2.1. The assumptions in Theorem 2.2 that D is bounded and all $f_i(D)$ contain a fixed neighborhood of $0 \in \mathbb{C}^n$ are necessary. In fact, the result in Theorem 2.2 does not hold without any one of the assumptions. For example, taking $f_i(z_1, z_2) = (z_1, z_2/i)$ as a sequence holomorphic maps from $D = \mathbb{C}^2$ to itself, the limit map $f : D \to \mathbb{C}^2$ is given by $f(z_1, z_2) = (z_1, 0)$, it is not injective even f_i are injective and $\mathbb{C}^2 \subset f_i(D)$ for all i; the restrictions $f_i|_{B^2}$ give a sequence of injective holomorphic maps from B^2 to $\mathbb{C}^2$ with limit map $f|_{B^2}$, which is not injective since not all $f_i(B^2)$ contain a fixed neighborhood of $0 \in \mathbb{C}^2$. On the other hand, by a result in [7], the two assumptions can be replaced by assuming that $|det J_{f_i}(x)|$ have a positive lower bound.

Using the results in Theorem 2.2 and Lemma 2.3, we give the proof of Theorem 2.1:

Proof. (Proof of Theorem 2.1) By definition of squeezing functions, there exist a sequence of holomorphic embeddings $f_i : D \to B^n$ with $f_i(x) = 0$, and a sequence of increasing positive numbers r_i convergent to $s_D(x)$ such that $B^n(0, r_i) \subset f_i(D)$. By Montel theorem, there exists a subsequence $\{f_{i_k}\}$ of f_i which converges to a homomorphic map $f : D \to \mathbb{C}^n$ uniformly on compact subsets of D. Since $B^n(0, r_1) \subset f_i(D)$ for all i, by Theorem 2.2, f is injective. In particular it is an open map and hence $f(D) \subset B^n$. Then we get a holomorphic embedding $f : D \to B^n$ with $f(0) = 0$.

To prove $B^n(0, s_D(x)) \subset f(D)$, it suffices to prove $B^n(0, r_j) \subset f(D)$ for each fixed integer j. By assumption, $B^n(0, r_j) \subset f_i(D)$ for all $i > j$. Let $g_i = f_i^{-1}|_{B^n(0, r_j)}$, then we have $f_{i_k} g_{i_k} = Id_{B^n(0, r_j)}$ for $i_k > j$. By Montel theorem, we may assume that the sequence $\{g_{i_k}\}$ converges to a holomorphic map $g : B^n(0, r_j) \to \mathbb{C}^n$ uniformly on compact subsets of $B^n(0, r_j)$. We want to prove that $g(B^n(0, r_j)) \subset D$. Note that $g(0) = x$, hence there exists a neighborhood U of 0 in $B^n(0, r_j)$ such that $g(U) \subset D$. This implies $f \cdot g|_U$ is defined and it is clearly equal to the identity map Id_U. So $det J_g(0) \neq 0$. Since $det J_{g_i} \neq 0$ for all $i > i_0$, by Lemma 2.3, we have $det J_f \neq 0$ and hence g is an open map, which implies that $g(B^n(0, r_j)) \subset D$.

Therefore $fg : B^n(0, r_j) \to B^n(0, r_j)$ is a well defined map. It is clear that $fg = Id_{B^n(0,r_j)}$, so we have $B^n(0, r_j) \subset f(D)$.

$\square$

3. CONTINUITY OF SQUEEZING FUNCTIONS

In this section, we will prove that the squeezing function on any bounded domain is continuous. As a consequence, a bounded domain is a holomorphic homogeneous regular domain if it covers a compact complex manifold.

Theorem 3.1. *The squeezing function s_D of any bounded domain D in $\mathbb{C}^n$ is continuous.*

Proof. Since D is a bounded domain, the Kobayashi metric on D is nondegenerate.

Let a be an arbitrary point in D and $\{z_k\}$ be a sequence in D convergent to a, and let ϵ be an arbitrary positive number. By Theorem 2.1, there exists a holomorphic embedding $f : D \to B^n$ such that $f(a) = 0$ and $B^n(0, s_D(a)) \subset f(D)$. Since f is continuous, there exists an integer N such that $||f(z_k)|| < \epsilon$ for $k > N$. Define $f_k : D \to \mathbb{C}^n$ as

$$f_k(z) = \frac{f(z) - f(z_k)}{1 + \epsilon}$$

for $k > N$, then $f_k(D) \subset B^n$, $f_k(z_k) = 0$ and

$$B^n(0, \frac{s_D(a) - \epsilon}{1 + \epsilon}) \subset f_k(D).$$

This implies that $s_D(z_k) \geq (s_D(a) - \epsilon)/(1 + \epsilon)$. Let ϵ tends to 0, we get

$$\liminf_{k \to \infty} s_D(z_k) \geq s_D(a).$$

Let $K_D(\cdot, \cdot)$ be the Kobayashi distance on D. It is known that K_D is continuous on $D \times D$ (see e.g. [9]). So we have $K_D(z_k, a) \to 0$ as $k \to \infty$. By Theorem 2.1, for each k, there exists a holomorphic embedding $f_k : D \to B^n$ such that $f_k(z_k) = 0$ and $B^n(0, s_D(z_k)) \subset f_k(D)$. By the decreasing property of Kobayashi distances (see e.g. [9]), we have

$$K_{B^n}(f_k(z_k), f_k(a)) = K_{B^n}(0, f_k(a)) \leq K_D(z_k, a)$$

for all k. So $K_{B^n}(0, f_k(a)) \to 0$, which implies that $f_k(a)$ tends to 0 in the ordinary topology (see [4]). So, for any positive number ϵ, there exists an integer M such that $||f_k(a)|| < \epsilon$ for $k > M$. This implies

$$s_D(a) \geq \frac{s_D(z_k) - \epsilon}{1 + \epsilon},$$

so we have

$$s_D(a) \geq \limsup_{k \to \infty} \frac{s_D(z_k) - \epsilon}{1 + \epsilon}.$$

Letting ϵ tends to 0, we get

$$s_D(a) \geq \limsup_{k \to \infty} s_D(z_k).$$

So $\lim_{k \to \infty} s_D(z_k) = s_D(a)$, namely s_D is continuous at a. Note that a is arbitrary, so s_D is continuous on D. $\square$

For $r \in [0, 1)$, we define

$$\sigma(r) = log \frac{1+r}{1-r}.$$

It is clear that $\sigma(c)$ is strictly increasing for $0 \leq c < 1$, its inverse is given by $\sigma^{-1}(w) = \tanh(w/2)$. For a point $z \in B^n$, the Kobayashi distance form 0 to z is $\sigma(|z|)$. For two nonnegative numbers u and v, it is not difficult to prove that

$$\sigma^{-1}(u + v) \leq \sigma^{-1}(u) + \sigma^{-1}(v).$$

Let D be a bounded domain in $\mathbb{C}^n$, we define a function $T(\cdot, \cdot)$ on $D \times D$ as

$$T(x, y) = \sigma^{-1}(K_D(x, y))$$

then the above properties of σ implies that $T(\cdot, \cdot)$ is a metric on D. Since K_D induces the ordinary topology of D, so does T. From the proof of Theorem 3.1, one can directly get the following

Theorem 3.2. *The squeezing function s_D of D is Lipschitz continuous with respect to the metric T. In fact, we have*

$$|s_D(x) - s_D(y)| \leq 2T(x, y), \ x, y \in D.$$

Remark 3.1. The same result as in the above theorem still holds if we replace Kobayashi distance in the definition of $T(\cdot, \cdot)$ by Carathéodory distance.

By Theorem 3.1, we directly get the following result proved in [21]:

Corollary 3.3. *Let D be a bounded domain that covers a compact complex manifold, then D is a holomorphic homogeneous regular domain.*

Proof. Let X be a compact complex manifold that is covered by D. By the holomorphic invariance of squeezing functions, $s_D(z)$ can be pushed down to a function on X. By Theorem 3.1, $s_D(z)$ is continuous. Note that $s_D(z)$ is also positive, it must attain a positive lower bound on X, and hence on D. $\qquad\square$

4. RELATIONS BETWEEN INTRINSIC METRICS AND SQUEEZING FUNCTIONS

The main purpose of this section is to investigate relations between squeezing functions and some intrinsic metrics on bounded domains. We give a boundary estimate of the Carathéodory metric of a bounded planar domain in term of boundary behavior of its squeezing function. In fact, similar but weaker form of this result still holds in higher dimensional case. We then focus on bounded holomorphic homogeneous regular domains, and prove that the Carathéodory metric, the Kobayashi metric and the Bergman metric on these domains are complete. As a result, a holomorphic homogeneous regular domain must be pseudoconvex.

We first need the following lemma, which is known as Koebe's one-quarter theorem in classical complex analysis.

Lemma 4.1. *(see e.g. [2]) Let $\Delta \subset \mathbb{C}$ be the unit disc. Let g be a univalent holomorphic function on Δ such that $g(0) = 0$ and $g'(0) = 1$. Then $\Delta_{1/4} := \{z \in \mathbb{C}; |z| < 1/4\} \subset g(\Delta)$.*

With this lemma, we now prove the following

Theorem 4.2. *Let D be a bounded domain in $\mathbb{C}$, $x \in D$. Then the Carathéodory pseudo-norm of $\frac{\partial}{\partial z}$ at x is not less than $s_D(x)/4\delta(x)$, where z is the standard coordinate on $\mathbb{C}$ and $\frac{\partial}{\partial z}$ is viewed as a vector in the tangent space $T_x D$ of D at x.*

Proof. By Theorem 2.1, there exists a univalent map $f : D \to \Delta$ such that $f(x) = 0$ and $\Delta_{s_D(x)} \subset f(D)$, where $\Delta_{s_D(x)}$ is the disc in $\mathbb{C}$ with center 0 and radius $s_D(x)$. We want to estimate the module $|f'(x)|$ of the derivative of f at x. Let

$$g = f^{-1}|_{\Delta_{s_D(x)}},$$

it is a univalent map form $\Delta_{s_D(x)}$ to D such that $g(0) = x$.

Now we define a univalent map $\varphi : \Delta \to \mathbb{C}$ by setting

$$\varphi(z) = \frac{g(s_D(x) \cdot z) - x}{s_D(x) \cdot g'(0)},$$

then it is clear that $\varphi(0) = 0$ and $\varphi'(0) = 1$. By Lemma 4.1, we have $\Delta_{1/4} \subset \varphi(\Delta)$. This implies that

$$\Delta(x, s_D(x)|g'(0)|/4) \subset g(\Delta_{s_D(x)}) \subset D,$$

where, for $a \in \mathbb{C}$ and $r > 0$, we set $\Delta(a, r)$ the disc in $\mathbb{C}$ with center a and radius r. In particular, we have

$$\delta(x) \geq s_D(x)|g'(0)|/4.$$

Note that $f'(x) = 1/g'(0)$, we get

$$|f'(x)| \geq \frac{s_D(x)}{4\delta(x)}.$$

This means that the Carathéodory pseudo-norm of $\frac{\partial}{\partial z}$ at x is not less than $s_D(x)/4\delta(x)$. $\qquad\square$

Remark 4.1. Using similar argument as in the proof of Theorem 4.2, one can prove a weaker form of Theorem 4.2 in higher dimensional cases. In fact, for a bounded domain $D \subset \mathbb{C}^n$, one can prove that the Carathéodory pseudo-norm $\| X \|_{C_D}$ on D of $X \in T_x D = \mathbb{C}^n$ admits the estimate

$$\| X \|_{C_D} \geq \frac{s_D(x) \| X \|}{4\delta(x, X)},$$

where $\delta(x, X)$ is the boundary distance of x with respect to the direction X, and $\| X \|$ is the Euclidean norm of X.

A corollary of theorem4.2 is the following

Theorem 4.3. *Let D be a bounded domain in $\mathbb{C}$ satisfying*

$$s_D(x) > C/log(1/\delta(x))$$

for some positive constant C and all $x \in D$ with $\delta(x) < 1$, then the Carathéodory metric on D is complete.

Proof. By Theorem 4.2, we see that the Carathéodory pseudo-norm of $\frac{\partial}{\partial z}$ at x is not less than $\frac{-C}{\delta(x)log\delta(x)}$, which implies the completeness of the Carathédory metric on D. $\qquad\square$

We focus on holomorphic homogeneous regular domains in the rest of this section. We first prove the completeness of the Carathéodory metric of a holomorphic homogeneous regular domain.

Theorem 4.4. *Let D be a holomorphic homogeneous regular domain in $\mathbb{C}^n$. Then the Carathéodory metric on D is complete.*

Proof. Since D is a bounded domain, the Carathéodory metric on D is nondegenerate. Denote by $C_D(\cdot, \cdot)$ the Carathéodory distance on D.

Let $c > 0$ be a positive lower bound of the squeezing function s_D of D. We first prove that, for any $x_0 \in D$, the set

$$A(x_0) := \{x \in D; C_D(x, x_0) < \log(\frac{1 + c/2}{1 - c/2})\}$$

is relatively compact in D. By theorem2.1, there exists an open holomorphic embedding $f_{x_0} : D \to B^n$ such that $f_{x_0}(x_0) = 0$ and $B^n(0, c) \subset f_{x_0}(D)$. By the decreasing property of Carathéodory metrics, we have

$$f_{x_0}^* C_{B^n} \leq C_D$$

this implies that

$$f_{x_0}(A(x_0)) \subset \{z \in B^n; C_{B^n}(z, 0) < \log(\frac{1 + c/2}{1 - c/2})\}.$$

Note that the set

$$\{z \in B^n; C_{B^n}(z, 0) < \log(\frac{1 + c/2}{1 - c/2})\} = \{z \in B^n; ||z|| < c/2\}$$

is relatively compact in $B^n(0, c)$, so $A(x_0)$ is relatively compact in D.

For bounded domains, it is known that the topology induced by the Carathéodory metric coincides with the ordinary topology(see e.g. [8]). In particular, a compact set of D with the ordinary topology is also compact with respect to the topology induced by the Carathéodory metric. By the above result, we see that the Carathéodory metric on D is complete. $\square$

It is known that a domain whose Carathéodory metric is complete must be pseudoconvex (see e.g. [8]), as a consequence of Theorem 4.4, we have the following result, which was proved in [21] by different method:

Corollary 4.5. *A holomorphic homogeneous regular domain must be pseudoconvex.*

For any complex manifold, it is well known that its Carathéodory pseudo-metric is always dominated by its Kobayashi pseudo-metric (see e.g. [9]). For bounded domains, a famous result of Lu in [16] says that the Carathéodory metric is always dominated by the Bergman metric. Note also that, for a bounded domain D, any one of the three intrinsic metrics- the Carathéodory metric, the Kobayashi metric, and the Bergman metric- induces the same topology as the ordinary one. Therefore, as a consequence of Theorem 4.4, we have the following

Theorem 4.6. *Let D be a holomorphic homogeneous regular domain. Then the Kobayashi metric and the Bergman metric on it are complete.*

5. Squeezing functions on planar domains

In this section, we will consider squeezing functions on planar domains. For finitely connected planar domains, we get a neat description of the boundary behavior of their squeezing functions. As a result, we get the necessary and sufficient condition for such a domain to be a holomorphic homogeneous regular domain. If D has smooth boundary, then

$$\lim_{z \to \partial D} s_D(z) = 1.$$

By continuity of s_D, this implies D is a holomorphic homogeneous regular domain. As a consequence, we can recover some important results about planar domains, for example, the three intrinsic metrics-the Carathéodory metric, the Kobayashi metric, and the Bergman metric-on a bounded planar domain with smooth boundary are all complete, and they are equivalent; all smoothly bounded planar domains are hyperconvex , i.e., they admit bounded exhaustive subharmonic functions. We also give a class of holomorphic homogeneous regular domains which are infinitely connected.

It is clear that the squeezing function of the unit disc is the constant function with value 1. By Riemann mapping theorem and holomorphic invariance of squeezing functions, the squeezing function of any simply connected bounded planar domain is also constant with value 1.

Now we consider squeezing functions on 2-connected planar domains. Define

$$A_r = \{z \in \mathbb{C}; r < |z| < 1\}$$

for $0 \le r < 1$. When $r > 0$, we call A_r an annulus. It is well-known that a 2-connected domain in $\mathbb{C}$ which is not conformal equivalent to $\mathbb{C}^*$ must be holomorphic equivalent to a unique A_r (see e.g. [1]). When $r = 0$, A_0 is just the punctured disc Δ^*, and we will consider it in the last section. For A_r with $r > 0$, we have the following

Theorem 5.1. *For $r > 0$, the squeezing function $s_{A_r}(z)$ tends to 1 as $z \to \partial A_r$. In particular, A_r is a holomorphic homogeneous regular domain.*

Proof. For $c \in [0,1)$, we define

$$\sigma(c) = log\frac{1+c}{1-c}.$$

It is clear that $\sigma(c)$ is strictly increasing for $0 \le c < 1$, its inverse is given by $\sigma^{-1}(w) = \tanh(w/2)$. For a point $z \in \Delta$, the Poincaré distance form 0 to z is $\sigma(|z|)$.

Now let $z \in A_r$, with respect to the Poincaré metric on Δ, the distance from z to the cycle $\{w; |w| = r\}$ is $\sigma(|z|) - \sigma(r)$. Denote by $P(z, \sigma(|z|) - \sigma(r))$ the disc (w.r.t the Poincaré metric on Δ) with center z and radius $\sigma(|z|) - \sigma(r)$, then we have $P(z, \sigma(|z|) - \sigma(r)) \subset A_r$. Choose a conformal map $f : \Delta \to \Delta$ such that $f(z) = 0$. Since f preserves the Poincaré metric on Δ, it maps $P(z, \sigma(|z|) - \sigma(r))$ onto the disc (w.r.t the Poincaré metric on Δ) with center $f(z) = 0$ and radius $\sigma(|z|) - \sigma(r)$, which is just the Euclidean disc with center 0 and radius $\sigma^{-1}(\sigma(|z|) - \sigma(r))$. This implies that

$$s_{A_r}(z) \ge \sigma^{-1}(\sigma(|z|) - \sigma(r)). \tag{2}$$

Note that $\sigma^{-1}(\sigma(|z|) - \sigma(r)) \to 1$ as $|z| \to 1$, so $s_{A_r}(z)$ tends to 1 as $|z| \to 1$. Consider the holomorphic automorphism of A_r given by $z \mapsto r/z$, by the conformal invariance of s_{A_r}, we also get $s_{A_r}(z)$ tends to 1 as $|z| \to r$.

By Theorem 3.1, s_{A_r} is continuous and hence has a positive lower bound, so A_r is a holomorphic homogeneous regular domain. $\qquad\square$

By similar argument, Theorem 5.1 can be generalized to finitely connected planar domains as follows:

Theorem 5.2. *Let D be a domain in $\mathbb{C}$, assume that $\bar{\mathbb{C}} - D$ has finitely many connected components such that each connected component is not a single point. Then we have*

$$\lim_{z \to \partial D} s_D(z) = 1.$$

In particular, D is a holomorphic homogeneous regular domain.

Proof. We define the function σ as in the proof of Theorem 5.1. Let $E_1, \cdots, E_n$ be connected components of $\bar{\mathbb{C}} - D$. Then $D_1 := \bar{\mathbb{C}} - E_1$ is simply connected and $D \subset D_1$. Since E_1 is not a single point, by Riemann mapping theorem, there is a conformal map φ_1 from D_1 to Δ. It is clear that $\varphi_1(D)$ is a domain in Δ obtained by deleting $n-1$ connected compact subsets, say $L_2, \cdots, L_n$, from Δ.

Let $z \in D$ and let $l_z = P_\Delta(z, \cup_{i=2}^n L_i)$ be the distance from z to $\cup_{i=2}^n L_i$ with respect to the Poincare distance of Δ. Choose a conformal map $f : \Delta \to \Delta$ such that $f(z) = 0$, then the Euclidean disc with center 0 and radius $\sigma^{-1}(l_z)$ is contained in $f(\varphi(D))$, which implies that $s_{\varphi(D)}(z) \geq \sigma^{-1}(l_z)$. If $|z|$ tends to 1, then l_z tends to ∞ and $\sigma^{-1}(l_z)$ tends to 1, hence $s_{\varphi(D)}(z)$ tends to 1. By holomorphic invariance of squeezing functions, we see that

$$\lim_{z \to E_1} s_D(z) = 1.$$

Similarly, for E_i with $2 \leq i \leq n$, we also have

$$\lim_{z \to E_i} s_D(z) = 1.$$

Hence

$$\lim_{z \to \partial D} s_D(z) = 1.$$

By continuity of s_D, it admits a positive lower bound on D, so D is a holomorphic homogeneous regular domain. $\square$

Using Riemann mapping theorem, one can prove that the domains considered in the above theorem are holomorphic equivalent to bounded domains with smooth boundary(see e.g. [1]). Hence an equivalent version of the above theorem is the following:

Theorem 5.3. *Let D be a bounded domain in $\mathbb{C}$ with smooth boundary. Then we have*

$$\lim_{z \to \partial D} s_D(z) = 1$$

In particular, D is a holomorphic homogeneous regular domain.

Remark 5.1. As a consequence of Theorem 5.3, for a bounded planar domain D with smooth boundary, s_D can be extended continuously to $\bar{D}$. It may be interesting to investigate whether the same result holds in higher dimensions.

In section 3, we have shown that the Carathéodory metric, the Kobayashi metric, and the Bergman metric on a holomorphic homogeneous regular domain are complete. As mentioned in the introduction, these intrinsic metrics on a holomorphic homogeneous regular domain are equivalent [14]. It is also turned out that any holomorphic homogeneous regular domain is hyperconvex [21]. So, as a result of Theorem 5.3, we can recover the following results in complex analysis:

Theorem 5.4. *Let D be a planar domain with smooth boundary, then we have:*
(1) The Carathéodory metric, the Kobayashi metric and the Bergman metric on D
are complete;
(2) The Carathéodory metric , the Kobayashi metric and the Bergman metric on
D are equivalent.
(3) D is hyperconvex.

By definition, it is clear that the product of two holomorphic homogeneous regular domains is again a holomorphic homogeneous regular domain. So we get

Corollary 5.5. *Let D be a bounded domain in $\mathbb{C}^n$ which is holomorphic equivalent to the product of bounded planar domains with smooth boundary, then D is a holomorphic homogeneous regular domain.*

As mentioned in the introduction, the list of known holomorphic homogeneous regular domains contains bounded homogeneous domains , Teichmüller spaces, bounded domains covering compact Kähler manifolds and strictly convex domains with C^2-boundary. Many examples of holomorphic homogeneous regular domains given by Corollary 5.5 are not in the list . More precisely, we have the following

Proposition 5.6. *Let $D_1, \cdots, D_k$ be bounded planar domains with smooth boundaries which are mutually not conformal equivalent. If there exists a D_i which is not conformal equivalent to the unit disc Δ, then the domain $D := D_1^{r_1} \times \cdots \times D_k^{r_k}$ is a holomorphic homogeneous regular domain which is not holomorphic equivalent to any of the domains in the above list. Where $r_1, \cdots, r_k$ are positive integers and $D_i^{r_i} = D_i \times \cdots \times D_i$ is the r_i-power of D_i.*

Proof. Denote by $Aut(D)$ the holomorphic automorphism group of D. By the proposition in [19] and Theorem 1 in [20], we have

$$Aut(D) = Aut(D_1^{r_1}) \times \cdots \times Aut(D_k^{r_k}),$$

and, for each i, the following sequence is exact:

$$1 \to (Aut(D_i))^{r_i} \to Aut(D_i^{r_i}) \to S_{r_i} \to 1,$$

where S_{r_i} is the symmetry group of degree r_i which acts on $D_i^{r_i}$ by permutation. The decomposition of $Aut(D)$ implies that D is homogeneous if and only if each factor D_i is homogeneous, and D can cover a compact complex manifold is and only if so does for each D_i. Note that a smoothly bounded planar domain is homogeneous or can cover a compact complex manifold if and only if it is isomorphic to Δ, hence D can not be homogeneous or cover a compact complex manifold.

It is clear that D can not be holomorphic equivalent to any convex domain since the fundamental group of D is nontrivial. By the same reason, D is not holomorphic equivalent to any Teichmüller space since it is well known that all Teichmüller spaces are contractible.   □

For general finitely connected planar domains, the boundary behavior of their squeezing functions can be described as follows:

Theorem 5.7. *Let D be a finitely connected planar domain, let E be a connected component of $\bar{\mathbb{C}} - D$:*
(1) if E is not a single point, then

$$\lim_{D \ni z \to E} s_D(z) = 1;$$

(2) if $E = \{p\}$ contains a single point, then

$$s_D(z) \leq \sigma^{-1}(K_{\tilde{D}}(z,p)), \quad z \in D,$$

where σ is defined as in the proof of Theorem 5.1 and $K_{\tilde{D}}(\cdot, \cdot)$ is the Kobayashi distance on the domain $\tilde{D} := D \cup \{p\}$. In particular,

$$\lim_{D \ni z \to p} s_D(z) = 0.$$

Proof. The proof of (1) is similar to the proof of Theorem 5.2 and we will not repeat it here. Now we give the proof of (2). Let $z \in D$, and $f_z : D \to \Delta$ be a holomorphic embedding such that $f_z(z) = 0$. Note that $\tilde{D}$ is a domain. By Riemann's removable singularity theorem, f_z can be extended as a holomorphic map $\tilde{f}_z$ form $\tilde{D}$ to Δ. It is clear that $\tilde{f}_z(p) \notin f(D)$. By the decreasing property of Kobayashi distances, we have $K_\Delta(0, \tilde{f}_z(p)) \leq K_{\tilde{D}}(z,p)$, hence $s_D(z) \leq \sigma^{-1}(K_{\tilde{D}}(z,p))$. $\square$

Remark 5.2. Theorem 5.7 implies that a finitely connected planar domain is a holomorphic homogeneous regular domain if and only if each connected component of its complement in $\bar{\mathbb{C}}$ is not a single point.

The examples of planar holomorphic homogeneous regular domains given by Theorem 5.2 are all finitely connected, i.e, their complement in $\bar{\mathbb{C}}$ have finitely many connected components. We can also construct a class of planar holomorphic homogeneous regular domains which are infinitely connected.

Let $Aut(\Delta)$ be the holomorphic automorphism group of Δ, and denote by Δ_r the disc in $\mathbb{C}$ with center 0 and radius r and $\overline{\Delta}_r$ its closure. We first prove the following lemma:

Lemma 5.8. *For any positive numbers u, v and w with $u < v < w < 1$, there exists a positive number $c(u,v,w)$ such that: for an arbitrary positive number r with $u < r < v$, let $D \subset A_r$ be a domain containing $\Delta_w \setminus \overline{\Delta}_r$, then, for $z \in \Delta_v \setminus \overline{\Delta}_r$, we have $s_D(z) \geq c(u,v,w)$.*

Proof. Consider the reflection $R : D \to \Delta$ given by

$$z \mapsto r/z,$$

It is clear that $z \in R(D)$ provided $r/w < |z| < 1$. For $z \in \Delta_v \setminus \overline{\Delta}_r$, we have

$$r/v < |R(z)| < 1,$$

hence the disc (with respect to the Poincare metric) with center z and radius $\sigma(r/v) - \sigma(r/w)$ is contained in $R(D)$. So we see that

$$s_{R(D)}(R(z)) < \sigma^{-1}(\sigma(r/v) - \sigma(r/w))$$

for $z \in \Delta_v \setminus \overline{\Delta}_r$. By the biholomorphic invariance of squeezing functions, we get

$$s_D(z) \geq \sigma^{-1}(\sigma(r/v) - \sigma(r/w))$$

for $z \in \Delta_v \setminus \overline{\Delta}_r$. Take

$$c(u,v,w) = \inf_{u \leq r \leq v} \{\sigma^{-1}(\sigma(r/v) - \sigma(r/w))\},$$

it is clear that $c(u,v,w) > 0$ and it satisfies the condition of the lemma. $\square$

By the above lemma, we can prove the following

Theorem 5.9. *Let u, v and w be positive numbers with $u < v < w < 1$, let r_k, $k = 1, 2, \cdots$, be a sequence of positive numbers satisfying $u < r_k < v$. Let f_k be a sequence in $Aut(\Delta)$ such that $f_k(\overline{\Delta}_w)$ are pairwise disjoint. Then the domain*

$$D = \Delta \setminus (\cup_{k=1}^{\infty} f_k(\overline{\Delta}_{r_k}))$$

is a holomorphic homogeneous regular domain.

Proof. Let $c(u, v, w)$ as in Lemma 5.8, denote $c(u, \frac{v+w}{2}, w)$ by c. By the above lemma and biholomorphic invariance of squeezing functions, we have $s_D(z) \geq c$ for

$$z \in D' := D \cap \left(\cup_{k=1}^{\infty} f_k(\Delta_{\frac{v+w}{2}}) \right).$$

For $z \in D \setminus D'$, the distance form z to ∂D with respect to the Poincare distance on Δ is greater than $\sigma(\frac{v+w}{2}) - \sigma(v)$. Take a conformal map $f \in Aut(\Delta)$ such that $f(z) = 0$, we see that $s_D(z) \geq \sigma^{-1}(\sigma(\frac{v+w}{2}) - \sigma(v))$. So s_D has a positive lower bound and hence D is a holomorphic homogeneous regular domain. $\square$

As a consequence, the Carathéodory metrics, the Kobayashi metrics, and the Bergman metrics on domains that are constructed in Theorem 5.9 are complete, and they are equivalent. An explicit example can be constructed as follows: take $f \in Aut(\Delta)$ defined by

$$f(z) = \frac{z + 1/2}{1 + z/2}$$

and let $D = \Delta \setminus (\cup_{k=-\infty}^{\infty} f^k(\overline{\Delta}_{1/4}))$, then D is a holomorphic homogeneous regular domain, and the Carathodory metric, the Kobayashi metric, and the Bergman metric on D are complete and equivalent. The special domain D was constructed in [10] to show that a bounded planar domain may have infinite discrete automorphism group.

6. SQUEEZING FUNCTIONS ON ANNULI

In the above section, we have studied some properties of squeezing functions of annuli. In this section, we want to investigate further properties of them.

We have seen that all annuli are holomorphic homogeneous regular domains, and their squeezing functions tend to 1 at the boundary. An interesting but difficult problem is to give an exact expression of s_{A_r}. Another relatively simple problem is to determine the minimum of s_{A_r} for $r > 0$, which are conformal invariants.

By the conformal invariance of s_{A_r}, $s_{A_r}(z)$ depends only on $|z|$, so s_{A_r} is reduced to a function defined on $(r, 1)$. Be the reflection $z \mapsto r/z$, it can be further reduced to a function on $[\sqrt{r}, 1)$. We show that $s_{A_r}(x)$ is strictly increasing on $[\sqrt{r}, 1)$. To prove this result, we first prove two propositions, which are also interesting in their own right.

Let $D \subset \mathbb{C}$ be a bounded domain with smooth boundary, denote by H_D the set of univalent maps f form D to Δ such that $\Delta \setminus f(D)$ is a compact set. For bounded planar domain D, we always denote by $P_D(\cdot, \cdot)$ the Poincare distance of D.

Proposition 6.1. *Let $D \subset \mathbb{C}$ be a bounded domain with smooth boundary. Then, for any $p \in D$, we have*

$$s_D(p) = \sup\{r | \Delta_r \subset f(D) \; for some \; f \in H_D \; with \; f(p) = 0\}.$$

Proof. Let $f : D \to \Delta$ be a univalent map with $f(p) = 0$ and assume $\Delta_r \subset f(D)$. Let D' be the union of $f(D)$ and the compact connected components of $\Delta \setminus f(D)$. Then D' is simply connected. By the Riemann mapping theorem, there is a conformal map $g : D' \to \Delta$ with $g(0) = 0$.

Note that

$$\Delta_r = \{z \in \Delta | P_\Delta(z, 0) < \sigma(r)\},$$

by the decreasing property of the Poincare metrics on planar domains, we have $P_\Delta(z, w) \leq P_{D'}(z, w)$ for all $z, w \in D'$, so

$$\{z \in D' | P_{D'}(z, 0) < \sigma(r)\} \subset \Delta_r.$$

Note that g is an isometry form $(D', P_{D'})$ to (Δ, P_Δ). Hence

$$\Delta_r = \{z \in \Delta | P_\Delta(z, 0) < \sigma(r)\} = g(\{z \in D' | P_{D'}(z, 0) < \sigma(r)\}) \subset g(f(D)).$$

Now we get a univalent map $g \circ f : D \to \Delta$ with $g \circ f \in H_D$, $g \circ f(p) = 0$ and $\Delta_r \subset g \circ f(D)$. Since f is arbitrary, we see that

$$s_D(p) = \sup\{r | \Delta_r \subset h(D) \; for \; some \; h \in H_D \; with \; h(p) = 0\}.$$

$\square$

For two subsets A and B of Δ, we denote by $P_\Delta(A, B)$ the distance between A and B with respect to the Poincare distance of Δ.

Proposition 6.2. *Let $D \subset \mathbb{C}$ be a bounded domain with smooth boundary. For $p \in D$. Let*

$$u(p) = \sup\{P_\Delta(f(p), \Delta \setminus f(D)) | f \in H_D\},$$

then we have $s_D(p) = \sigma^{-1}(u(p))$.

Proof. For $p \in D$ and $f \in H_D$, let

$$u = P_\Delta(f(p), \Delta \setminus f(D)).$$

Then the P_Δ-disc with center $f(p)$ and radius u is contained in $f(D)$. Take a conformal map $g : \Delta \to \Delta$ such that $g(f(p)) = 0$, then we have $\Delta_{\sigma^{-1}(u)} \subset g \circ f(D)$. So we have $s_D(p) \geq \sigma^{-1}(u)$. Since $f \in H_D$ is arbitrary, we have $s_D(p) \geq \sigma^{-1}(u(p))$.

On the other hand, for an arbitrary $f \in H_D$ with $f(p) = 0$, let r be the positive number such that $\Delta_r \in f(D)$ but $\Delta_{r+\epsilon} \nsubseteq f(D)$ for any $\epsilon > 0$. It is clear that

$$r = \sigma^{-1}\left(P_\Delta(f(p), \Delta \setminus f(D))\right) \leq \sigma^{-1}(u(p)).$$

Since $f \in H_D$ is arbitrary, by the above proposition, we have $s_D(p) \leq \sigma^{-1}(u(p))$.
$\square$

Theorem 6.3. *Viewed as a function on $[\sqrt{r}, 1)$, the squeezing function $s_{A_r}(z)$ of A_r is strictly increasing on $[\sqrt{r}, 1)$; in particular, it attains its minimum at $\sqrt{r}$.*

Proof. For simplicity, let $s = s_{A_r}$. For $x \in [\sqrt{r}, 1)$, by Theorem2.1, there is a univalent map $f : D \to \Delta$ such that $f(x) = 0$ and $\Delta_{s(x)} \subset f(D)$. By Proposition6.1, we may assume $f \in H_D$. By proposition6.2, we have $s(x) = \sigma^{-1}(P_\Delta(f(x), \Delta \setminus f(D)))$. By proposition6.2 and the conformal invariance of s, we have the identity

$$P_\Delta(f(x), \Delta \setminus f(D)) = sup\{P_\Delta(f(z), \Delta \setminus f(D)) | z \in D, |z| = x\}.$$

Note that the curve $f(|z| = \sqrt{r})$ separates $\mathbb{C}$ into two connected parts, let U and V be the bounded and unbounded connected components of $\mathbb{C} \setminus f(|z| = \sqrt{r})$ respectively.

If $x > \sqrt{r}$, then $f(|z| = x) \subset V$. In fact, if it is not the case, then composing f with the reflection $z \mapsto r/z$ will lead to a contradiction to the extremal property assumption on f.

Now let $x' \in [\sqrt{r}, 1)$ with $x' > x$, then it is clear that $f(|z| = x')$ lies in the unbounded component of $\mathbb{C} \setminus f(|z| = x)$, so we have

$$sup\{P_\Delta(f(z), \Delta \setminus f(D)) | z \in D, |z| = x'\} > sup\{P_\Delta(f(z), \Delta \setminus f(D)) | z \in D, |z| = x\}$$

by proposition6.2, there is a point $z \in A_r$ with $|z| = x'$ and $s_{A_r}(z) > s(x)$. Note that $s_{A_r}(z) = s(|z|) = s(x')$, hence $s(x') > s(x)$. $\square$

Theorem 6.3 and its proof lead us to conjecture that, for $\rho \in [\sqrt{r}, 1)$, $s_{A_r}(\rho)$ is given by

$$s_{A_r}(\rho) = \sigma^{-1}(\sigma(\rho) - \sigma(r)) = \sigma^{-1}\left(\log \frac{(1+\rho)(1-r)}{(1-\rho)(1+r)}\right),$$

where the function σ is defined as in the proof of Theorem 5.1. Provided this conjecture, Theorem 6.3 implies that $s_{A_r}(\rho)$ attains its minimum

$$s_{A_r}(\sqrt{r}) = \tanh \log \frac{1 + \sqrt{r}}{\sqrt{1+r}}$$

at $\rho = \sqrt{r}$, which characterizes the conformal structure of A_r.

7. Explicit form of squeezing functions on some special domains

In this section, we give the explicit form of squeezing functions on some special domains, namely punctured balls and classical bounded symmetric domains.

We first consider domains constructed by deleting analytic subsets from other domains.

Theorem 7.1. *Let $D' \subset \mathbb{C}^n$ be a bounded domain and $A \subset D'$ be a proper analytic subset. Then, for the domain $D = D' \setminus A$, one has*

$$s_D(z) \le \sigma^{-1}(K_{D'}(z, A)), \quad z \in D$$

where σ is defined as in the proof of Theorem 5.1 and $K_{D'}(\cdot, \cdot)$ is the Kobayashi distance on D'; in particular, we have $\lim_{z \to z_0} s_D(z) = 0$.

Proof. Let $z \in D$, and $f_z : D \to B^n$ be a holomorphic open embedding such that $f_z(z) = 0$. By Riemann's removable singularity theorem, f_z can be extended to a holomorphic map $\tilde{f}_z$ form D' to B^n. It is clear that $\tilde{f}_z(D) \cap \tilde{f}_z(A) = \emptyset$. By the decreasing property of Kobayashi distance, we have $K_{B^n}(0, \tilde{f}_z(A)) \le K_{D'}(z, A)$, hence $s_D(z) \le \sigma^{-1}(K_{D'}(z, A)), \quad z \in D$. $\square$

Remark 7.1. The domains D constructed in the above theorem are not holomorphic homogeneous regular domains. The conclusion can also be derived from Corollary 4.5 since D is not pseudoconvex, or from Theorem 4.4 since the Carathéodory metric on D, which is just the restriction of that of D', is not complete.

For the special case of punctured balls, Theorem 7.1 implies the following

Corollary 7.2. *The squeezing function $s_{B^n \setminus \{0\}}$ on the n dimensional punctured ball $B^n \setminus \{0\}$ is given by*

$$s_{B^n \setminus \{0\}}(z) = \| z \|$$

where $\| z \|$ is the Euclidean norm of z.

Proof. By theorem7.1, we have $s_{B^n \setminus \{0\}}(z) \leq \| z \|$. On the other hand, it is clear that $s_{B^n \setminus \{0\}}(z) \geq \| z \|$. Hence $s_{B^n \setminus \{0\}}(z) = \| z \|$. □

Other examples of bounded domains whose squeezing functions can be given explicitly are classical symmetric bounded domains. Recall that a classical symmetric domain is a domain of one of the following four types:

$D_I(r,s) = \{Z = (z_{jk}) : I - Z\bar{Z}' > 0, \text{ where } Z \text{ is an } r \times s \text{ matrix}\}$ $(r \leq s)$,

$D_{II}(p) = \{Z = (z_{jk}) : I - Z\bar{Z}' > 0, \text{ where } Z \text{ is a symmetric matrix of order } p\}$,

$D_{III}(q) = \{Z = (z_{jk}) : I - Z\bar{Z}' > 0, \text{ where } Z \text{ is a skew-symmetric matrix of order } q\}$,

$D_{IV}(n) = \{z = (z_1, \cdots, z_n) \in \mathbb{C}^n : 1 + |zz'|^2 - 2zz' > 0, \ 1 - |zz'| > 0\}$.

Here I is the identity matrix of proper order, $\bar{Z}$ denotes the conjugate matrix of Z and Z' the transposed matrix of Z. The complex dimensions of these four domains are rs, $p(p+1)/2$, $q(q-1)/2$ and n respectively.

For a bounded homogeneous domain D, by the holomorphic invariance of squeezing functions, s_D is a constant function on D, and we denote this constant by $s(D)$. By a theorem of Kubota, which is based on an earlier work of Alexander [3], the squeezing functions on the above four types of domains can be given explicitly as follows:

Theorem 7.3. *(see Theorem 1 in* [12]*)*

$$s(D_I(r,s)) = r^{-1/2},$$
$$s(D_{II}(p)) = p^{-1/2},$$
$$s(D_{III}(q)) = [q/2]^{-1/2},$$
$$s(D_{IV}(n)) = 2^{-1/2},$$

where $[q/2]$ *denotes the integral part of* $q/2$.

For products of classical symmetric domains, we have:

Theorem 7.4. *(see Theorem 2 in* [12]*) If* $D_1, \cdots, D_m$ *are classical symmetric domains, then*

$$s(D_1 \times \cdots \times D_m) = (s(D_1)^{-2} + \cdots + s(D_m)^{-2})^{-1/2}.$$

Remark 7.2. In the 1980s, Y. Kubota considered the following Carathéodory extremal problem [11, 12, 13]:

$$M(z_0, D) = \sup_{F \in \mathfrak{F}(D)} |J_F(z_0)| \qquad (z_0 \in D), \tag{3}$$

where D is a bounded domain in the complex Euclidean space $\mathbb{C}^n$ and $\mathfrak{F}(D)$ consists of all holomorphic mappings from D into the unit ball B^n in $\mathbb{C}^n$, and J_F is the Jacobian of F.

He proved that the extremal mapping of the extremal problem (3) is unique up to a unitary linear transformation when D is a bounded symmetric domain(including two exceptional cases) [13]. We observe that the extremal mappings are exact the extremal embedding from bounded symmetric domains into the unit ball. Take $D_I(r,s)$ here for example, we can find from Kubota's proof that the extremal mapping is $f(z) = z/\sqrt{r}$, $z = (z_{11}, \cdots, z_{1s}, z_{21}, \cdots, z_{rs}) \in \mathbb{C}^{rs}$, which is exact an

extremal embedding for the squeezing function $s(D_I(r,s)) = r^{-1/2}$ since one knows $B_n \subset D_I(r,s) \subset \sqrt{r}B^n$, $(n = rs)$.

When D is a complex ellipsoid in $\mathbb{C}^n$, i.e.

$$D = D(k_1, \cdots, k_n) = \big\{ z \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j|^{k_j} < 1 \big\},$$

where $k_j(j = 1, 2, \cdots, n)$ are positive real numbers, Ma considered the extremal problem (3) in 1997 [17]. It is proved that the extremal mapping is again linear, and we conjecture it is likely the extremal embedding for squeezing function $s_D(z)$ in this case. Therefore, it will be interesting to consider relations in general between squeezing function s_D on a bounded domain D and the Carathéodory maps from D into the unit ball, especially when D is homeomorphic to a cell.

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